A TOPOLOGICALLY INDUCED 2-IN/2-OUT OPERATION ON LOOP COHOMOLOGY

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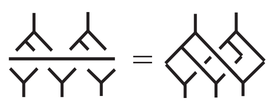
Abstract. We apply the Transfer Algorithm introduced in [7] to transfer an A_∞-algebra structure that cannot be computed using the classical Basic Perturbation Lemma. We construct a space X whose topology induces a nontrivial 2-in/2-out operation ω_2^2 on loop cohomology $H^*(\Omega X;\mathbb{Z}_2)$.

1. Introduction

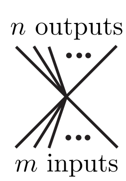
In [6] and [7], S. Saneblidze and this author defined the notions of a matrad and a relative matrad, and constructed the related families of polytopes known as biassociahedra $KK = \{KK_{n,m} = KK_{m,n}\}$ and bimultiplihedra $JJ = \{JJ_{n,m} = JJ_{m,n}\}$ of which $KK_{1,n}$ is the associahedron K_n and $JJ_{1,n}$ is the multiplihedron J_n. Cells of KK and JJ are identified with certain fraction product monomials, for example,



$\leftrightarrow$ a vertex of $KK_{2,3}$.

In fact, $JJ_{m,n}$ is a subdivision of $KK_{m,n} \times I$ with $\partial JJ_{m,n}$ containing the cells $KK_{n,m} \times 0$ and $KK_{n,m} \times 1$.

Let R be a commutative ring with unity. The free matrad $\mathcal{H}_\infty$ is represented by the DG R-module (DGM) of cellular chains $C_*(KK)$ by associating the top dimensional cell of $KK_{n,m}$ with the matrad generator $\theta_m^n \in \mathcal{H}_\infty$:



$\leftrightarrow \quad \theta_m^n$.

An A_∞-*bialgebra* is a DGM (A,d) together with a family of multilinear operations $\omega = \{\omega_m^n \in Hom^{m+n-3}(A^{\otimes m}, A^{\otimes n}) \mid mn \neq 1\}$ and a map of matrads $\mathcal{H}_\infty \to \mathcal{E}nd_{TA}$ such that $\theta_m^n \mapsto \omega_m^n$, i.e., (A,ω) is an algebra over $\mathcal{H}_\infty$. Note that we recover the operadic structure of A_∞-(co)algebras by setting $m = 1$ or $n = 1$.

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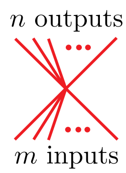
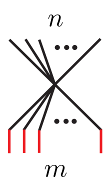
Similarly, the free relative matrad $\mathcal{JJ}_\infty$ is represented by the DGM of cellular chains $C_*(JJ)$ by associating the top dimensional cell of $JJ_{m,n}$ with the relative matrad generator $\mathfrak{f}_m^n \in \mathcal{JJ}_\infty$:

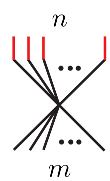


$$\leftrightarrow \quad \mathfrak{f}_m^n \ .$$

Let (A, ω_A) and (B, ω_B) be A_∞-bialgebras. A *morphism* G *from* A *to* B, denoted by $G : A \Rightarrow B$, is a family of multilinear maps $G = \{g_m^n \in Hom^{m+n-2}(A^{\otimes m}, B^{\otimes n})\}$ together with a map of relative matrads $\mathcal{JJ}_\infty \to Hom\,(TA, TB)$ such that $\mathfrak{f}_m^n \mapsto g_m^n$, i.e., G is an $\mathcal{H}_\infty$-bimodule. The elements $\theta_m^n \left(\mathfrak{f}_1^1\right)^{\otimes m}$ and $\left(\mathfrak{f}_1^1\right)^{\otimes n} \theta_m^n$ of $\mathcal{JJ}_\infty$ are associated with the codimension 1 cells $KK_{n,m} \times 0$ and $KK_{n,m} \times 1$ of $JJ_{m,n}$, respectively,



$$\leftrightarrow \quad \theta_m^n \left(\mathfrak{f}_1^1\right)^{\otimes m} \quad ;$$



$$\leftrightarrow \quad \left(\mathfrak{f}_1^1\right)^{\otimes n} \theta_m^n \ ,$$

and the aforementioned map of relative matrads sends $\theta_m^n \left(\mathfrak{f}_1^1\right)^{\otimes m} \mapsto \omega_m^n g^{\otimes m}$ and $\left(\mathfrak{f}_1^1\right)^{\otimes n} \theta_m^n \mapsto g^{\otimes n} \omega_m^n$. Again, we recover the structure of an A_∞-(co)algebra morphism by setting $m = 1$ or $n = 1$. A morphism $\Phi = \{\phi_m^n\}_{m,n \geq 1} : A \Rightarrow B$ is an *isomorphism* if ϕ_1^1 is an isomorphism of underlying modules.

The paper is organized as follows: In section 2 we review the *Transfer Algorithm* introduced in [7] and apply it to transfer an A_∞-algebra structure that cannot be computed using the classical Basic Perturbation Lemma. In Section 3 we construct a space X whose topology induces a nontrivial 2-in/2-out operation ω_2^2 on loop cohomology $H^*(\Omega X; \mathbb{Z}_2)$.

2. Transfer of A_∞-Structure

If A is a free DGM, B is an A_∞-algebra, and $g : A \to B$ is a homology isomorphism with a right-homotopy inverse, the Basic Perturbation Lemma (BPL) transfers the A_∞-algebra structure from B to A (see [3], [5], for example). When B is an A_∞-bialgebra, Theorem 1 generalizes the BPL in two directions:

(1) The A_∞-bialgebra structure on B transfers to an A_∞-bialgebra structure on A.
(2) The transfer algorithm requires neither freeness in A nor the existence of a right-homotopy inverse of g.

Given DGMs (A, d_A) and (B, d_B), let ∇ be the induced differential on $U_{A,B} = Hom\,(TA, TB)$, i.e., for $f \in U_{A,B}$ define $\nabla f = d_B f - (-1)^{|f|} f d_A$, where d_A and d_B denote the free linear extensions of d_A and d_B. A chain map $g : A \to B$ induces a cochain map $\tilde{g} : \mathcal{E}nd_{TA} \to U_{A,B}$ defined on $u \in Hom\,(A^{\otimes m}, A^{\otimes n})$ by $\tilde{g}\,(u) = g^{\otimes n} u$. If g is a homology isomorphism, so is $\tilde{g}$ provided condition (i) or (ii) in the following proposition is satisfied (the proof is left to the reader):

Proposition 1. *Let (A, d_A) and (B, d_B) be DGMs, and let $g : A \to B$ be a chain map that is also a homology isomorphism. Then $\tilde{g} : \mathcal{E}nd_{TA} \to U_{A,B}$ is a homology isomorphism if either of the following conditions holds:*

(i) A is free as an R-module.
(ii) For each $n \geq 1$, there is a DGM $X(n)$ and a splitting $B^{\otimes n} = A^{\otimes n} \oplus X(n)$ as a chain complex such that $H^* Hom\left(A^{\otimes k}, X(n)\right) = 0$ for all $k \geq 1$.

Thus there is the following generalization of the BPL:

Theorem 1 (The Transfer). *Let (A, d_A) be a DGM, let (B, d_B, ω_B) be an A_∞-bialgebra, and let $g : A \to B$ be a chain map and a homology isomorphism. If $\tilde{g} : \mathcal{E}nd_{TA} \to U_{A,B}$ is a homology isomorphism, then*

(i) (Existence) g induces an A_∞-bialgebra structure $\omega_A = \{\omega_A^{n,m}\}$ on A and extends to a map $G = \{g_m^n \mid g_1^1 = g\} : A \Rightarrow B$ of A_∞-bialgebras.
(ii) (Uniqueness) (ω_A, G) is unique up to isomorphism, i.e., if (ω_A, G) and $\left(\bar{\omega}_A, \bar{G}\right)$ are induced by chain homotopic maps g and $\bar{g}$, there is an isomorphism $\Phi : (A, \bar{\omega}_A) \Rightarrow (A, \omega_A)$ and a chain homotopy $T : \bar{G} \simeq G \circ \Phi$.

The proof of Theorem 1, which appears in [7], suggests the following general *Transfer Algorithm*:

The Transfer Algorithm

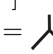
Initial data

- A DGM (A, d_A)
- An A_∞-bialgebra (B, d_B, ω_B) and a map of matrads $\alpha_B : C_*(KK) \to \mathcal{E}nd_{TB}$ sending $\theta_m^n \mapsto \omega_B^{n,m}$
- A chain map/homology isomorphism $g : A \to B$ such that $\tilde{g}$ is a homology isomorphism

Objectives

- Define operations $\omega_A^{n,m} : A^{\otimes m} \to A^{\otimes n}$ for all m, n, $mn \neq 1$
- Construct a map of matrads $\alpha_A : C_*(KK) \to \mathcal{E}nd_{TA}$ sending $\theta_m^n \mapsto \omega_A^{n,m}$
- Construct a map of A_∞-bialgebras $G = \left\{g_m^n \mid g_1^1 = g\right\} : A \Rightarrow B$

Initialization

1. Define $\beta : C_0(JJ_{1,1}) \to Hom(A, B)$ by $\mathfrak{f}_1^1 =$ $\mapsto g$
2. Define β on the vertex  of $JJ_{1,2}$ by $\theta_2^1\left(\mathfrak{f}_1^1 \otimes \mathfrak{f}_1^1\right) \mapsto \omega_B^{1,2}(g \otimes g)$
3. Define β on the vertex  of $JJ_{2,1}$ by $\theta_1^2 \mathfrak{f}_1^1 \mapsto \omega_B^{2,1} g$
4. Consider the ∇-cocycle $\omega_B^{1,2}(g \otimes g)$
    - Choose a cocycle $\omega_A^{1,2} \in \mathcal{E}nd_{TA}$ such that $\tilde{g}_*[\omega_A^{1,2}] = [\omega_B^{1,2}(g \otimes g)]$
    - Define $\alpha_A : C_0(KK_{1,2}) \to Hom\left(A^{\otimes 2}, A\right)$ by $\theta_2^1 =$  $\mapsto \omega_A^{1,2}$
    - Define $\alpha_A : C_0(\partial KK_{1,3}) \to Hom\left(A^{\otimes 3}, A\right)$ by

       $\mapsto \omega_A^{1,2}\left(\omega_A^{1,2} \otimes \mathbf{1}\right)$ and  $\mapsto \omega_A^{1,2}\left(\mathbf{1} \otimes \omega_A^{1,2}\right)$

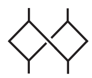
    - Extend β to the vertex  $\subset JJ_{1,2}$ via $\mathfrak{f}_1^1 \theta_2^1 \mapsto g\omega_A^{1,2}$
5. Dually, consider the ∇-cocycle $\omega_B^{2,1} g$
    - Choose a cocycle $\omega_A^{2,1} \in \mathcal{E}nd_{TA}$ such that $\tilde{g}_*[\omega_A^{2,1}] = [\omega_B^{2,1} g]$

- Define $\alpha_A : C_0\left(KK_{2,1}\right) \to Hom\left(A, A^{\otimes 2}\right)$ by $\theta_1^2 =$  $\mapsto \omega_A^{2,1}$
- Define $\alpha_A : C_0\left(\partial KK_{3,1}\right) \to Hom\left(A, A^{\otimes 3}\right)$ by

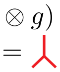
 $\mapsto \left(\omega_A^{2,1} \otimes \mathbf{1}\right)\omega_A^{2,1}$ and  $\mapsto \left(\mathbf{1} \otimes \omega_A^{2,1}\right)\omega_A^{2,1}$

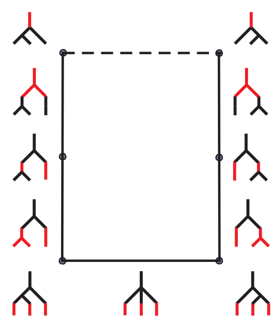
- Extend β to the vertex  $\subset JJ_{1,2}$ via $\left(\mathfrak{f}_1^1 \otimes \mathfrak{f}_1^1\right)\theta_1^2 \mapsto (g \otimes g)\,\omega_A^{2,1}$

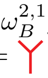
6. Define $\alpha_A : C_0\left(\partial KK_{2,2}\right) \to Hom\left(A^{\otimes 2}, A^{\otimes 2}\right)$ by

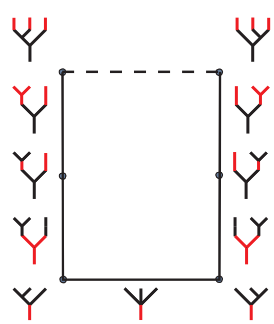
 $\mapsto \left(\omega_A^{1,2} \otimes \omega_A^{1,2}\right)\sigma_{2,2}\left(\omega_A^{2,1} \otimes \omega_A^{2,1}\right)$ and  $\mapsto \omega_A^{2,1}\omega_A^{1,2}$, where

$\sigma_{p,q} : \left(A^{\otimes p}\right)^{\otimes q} \overset{\approx}{\to} \left(A^{\otimes q}\right)^{\otimes p}$ is the canonical permutation of tensor factors

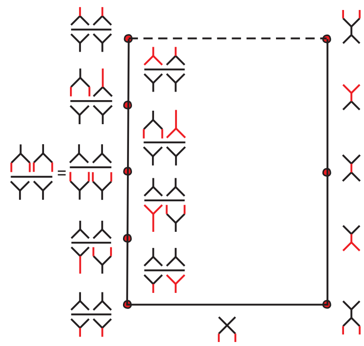
7. Note that $\left[\omega_B^{1,2}(g \otimes g) - g\omega_A^{1,2}\right] = 0$
   - Choose a cochain g_2^1 such that $\nabla g_2^1 = \omega_B^{1,2}(g \otimes g) - g\omega_A^{1,2}$
   - Define $\beta : C_1\left(JJ_{1,2}\right) \to Hom\left(A^{\otimes 2}, B\right)$ by $\mathfrak{f}_2^1 =$  $\mapsto g_2^1$
   - Define β on a monomial in $C_*\left(\partial JJ_{1,3} \smallsetminus \mathrm{int}KK_{1,3} \times 1\right)$ to be the corresponding composition:



8. Dually, note that $\left[\omega_B^{2,1}g - (g \otimes g)\,\omega_A^{2,1}\right] = 0$
   - Choose a cochain $g_1^2 \in U_{A,B}$ such that $\nabla g_1^2 = \omega_B^{2,1}g - (g \otimes g)\,\omega_A^{2,1}$
   - Define $\beta : C_1\left(JJ_{2,1}\right) \to Hom\left(A, B^{\otimes 2}\right)$ by $\mathfrak{f}_1^2 =$  $\mapsto g_1^2$
   - Define β on a monomial in $C_*\left(\partial JJ_{3,1} \smallsetminus \mathrm{int}KK_{3,1} \times 1\right)$ to be the corresponding composition:



9. Define β on a monomial in $C_*\left(\partial JJ_{2,2} \smallsetminus \mathrm{int}KK_{2,2} \times 1\right)$ to be the corresponding fraction product:



Induction hypothesis

Given $m+n \geq 4$, assume that for $i+j<m+n$, $ij \neq 1$, there exists a map

- $\alpha_A : C_*\left(KK_{j,i}\right) \to Hom\left(A^{\otimes i}, A^{\otimes j}\right)$ of matrads sending $\theta_i^j \mapsto \omega_A^{j,i}$
- $\beta : C_*\left(JJ_{j,i}\right) \to Hom\left(A^{\otimes i}, B^{\otimes j}\right)$ of relative matrads sending $\mathfrak{f}_i^j \mapsto g_i^j$

Induction objectives

- Define α_A on the generator $\theta_m^n \in C_{m+n-3}\left(KK_{n,m}\right)$
- Define β on the monomial $\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n \in C_{m+n-3}\left(JJ_{n,m}\right)$
- Define β on the generator $\mathfrak{f}_m^n \in C_{m+n-2}\left(JJ_{n,m}\right)$

Induction

For each $i+j=m+n$, $ij \neq 1$

1. Define α_A on each monomial in $C_*\left(\partial KK_{n,m}\right)$ to be its corresponding fraction product of operations in $\{\omega_A^{j,i}\}$; let
$$z=\alpha_A\left(C_{m+n-4}(\partial KK_{n,m})\right)$$
2. Define β on each monomial in $C_*\left(\partial JJ_{n,m} \smallsetminus \mathrm{int}KK_{n,m}\times 1\right)$ to be its corresponding fraction product of operations and maps in $\{\omega_A^{j,i}, g_i^j, \omega_B^{j,i}\}$; let
$$\varphi=\beta\left(C_{m+n-3}\left(\partial JJ_{n,m} \smallsetminus \mathrm{int}KK_{n,m}\times 1\right)\right)$$
3. Then $\tilde{g}\left(z\right)=\nabla\varphi$ implies $[z]=0$; choose a cochain b such that $\nabla b=z$
4. Note that $\nabla\left(\tilde{g}\left(b\right)-\varphi\right)=\nabla\tilde{g}\left(b\right)-\tilde{g}\left(z\right)=\tilde{g}\left(\nabla b-z\right)=0$; choose a cocycle
$$u\in\tilde{g}_*^{-1}\left[\tilde{g}\left(b\right)-\varphi\right]$$
5. Define $\alpha_A(\theta_m^n)=\omega_A^{n,m}:=b-u$
6. Define $\beta\left(\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n\right)=g^{\otimes n}\omega_A^{n,m}$
7. Note that $[\tilde{g}\left(\omega_A^{n,m}\right)-\varphi]=[\tilde{g}\left(b-u\right)-\varphi]=[\tilde{g}\left(b\right)-\varphi]-[\tilde{g}\left(u\right)]=0$; choose a cochain g_m^n such that
$$\nabla g_m^n=\tilde{g}\left(\omega_A^{n,m}\right)-\varphi=g^{\otimes n}\omega_A^{n,m}-\varphi$$
8. Define $\beta\left(\mathfrak{f}_m^n\right)=g_m^n$

This completes the induction.

Let us apply the Transfer Algorithm to compute an induced A_∞-algebra structure on the cohomology of a DGA, which cannot be computed using the classical BPL. The DGA B considered here has no Hodge decomposition, its homology

$H = H^*(B)$ is not free, and the given homology isomorphism $g : H \to B$ has no right-homotopy inverse.

Consider the tensor algebra $T^a M$ of the $\mathbb{Z}$-cochain complex

$$\begin{array}{ccccccccccccc} (M,d) & : & \mathbb{Z} & \to & 0 & \to & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \to & \mathbb{Z}_4 & \to & \mathbb{Z}_2 & \to & 0 \\ & & 1 & & & & (a,b) & & c & \mapsto & x & & \\ & & & & & & b & \mapsto & 2c & & & & \end{array}$$

and form the quotient $B = T^a M \big/ \left(a^2 + x, xc + cx, (ac+ca)^2, c^2, tb, bt\right)$, $|t| > 0$. Although the DGA B is not commutative, we do have $a\,(ac+ca) = (ac+ca)\,a$ and $(ac)^2 = (ca)^2$. Note that B has no Hodge decomposition since c is not a cocycle, $2c$ is a coboundary, and $\mathbb{Z}_4$ does not split as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Furthermore,

$$H^n(B) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}_2 & n = 2,5,7 \\ 0 & \text{otherwise} \end{cases}$$

and $H = H^*(B)$ is not free. Define $g : H \to B$ by $g(1) = 1$ and

$$\begin{array}{lcl} u = [a] & \mapsto & a \\ v = [ac+ca] & \mapsto & ac+ca \\ w = [a\,(ac+ca)] & \mapsto & a\,(ac+ca)\,. \end{array}$$

Then g has no right-homotopy inverse f since $gf(b) = \lambda a$ implies $b - \lambda a = (1-gf)(b) = (sd+ds)(b) = sd(b) = s(2c) = 2s(c) = 0$, which is a contradiction. To compute the induced multiplication μ_H, consider the following bases for H and $H \otimes H$:

	2	4	5	7	9	10	12	14
H	u		v	w				
$H \otimes H$		$u\vert u$		$u\vert v,\ v\vert u$	$u\vert w,\ w\vert u$	$v\vert v$	$v\vert w,\ w\vert v$	$w\vert w$

Ignoring the unit 1 and evaluating $\tilde{g}$ on the basis $\left\{w\partial_{u|v}, w\partial_{v|u}\right\}$ for $Hom^0\left(H^{\otimes 2}, H\right)$ we have

$$\tilde{g}\left(w\left(\partial_{u|v} + \partial_{v|u}\right)\right) = g(w)\left(\partial_{u|v} + \partial_{u|v}\right) = \mu\,(g \otimes g)\,.$$

Now thinking of $w\left(\partial_{u|v} + \partial_{v|u}\right)$ as a class in $H^*\left(Hom^*\left(H^{\otimes 2}, H\right)\right)$ we have

$$\tilde{g}_*\left[w\left(\partial_{u|v} + \partial_{v|u}\right)\right] = \left[g(w)\left(\partial_{u|v} + \partial_{u|v}\right)\right] = \left[\mu\,(g \otimes g)\right].$$

Define $\mu_H = (\tilde{g}_*)^{-1}\left[\mu\,(g \otimes g)\right] = w\left(\partial_{u|v} + \partial_{v|u}\right)$; then $uv = vu = w$ and μ_H is associative. To extend g to an $A(2)$-map, let μ denote the multiplication in B and consider the expression

$$\begin{aligned} z = \mu\,(g \otimes g) - g\mu_H &= a^2\partial_{u|u} + \left(a^3c + ca^3\right)\left(\partial_{u|w} + \partial_{w|u}\right) \\ &= dc\partial_{u|u} + d\,(cac)\left(\partial_{u|w} + \partial_{w|u}\right). \end{aligned}$$

Then $\nabla\left(c\partial_{u|u} + cac\left(\partial_{u|w} + \partial_{w|u}\right)\right) = z$ and we define $g_2 = c\partial_{u|u} + cac\left(\partial_{u|w} + \partial_{w|u}\right)$ so that

$$\nabla g_2 = \mu\,(g \otimes g) - g\mu_H\,.$$

Thus g is homotopy multiplicative. To compute the induced associator μ_H^3, consider the following bases for H and $H^{\otimes 3}$:

	2	5	6	7	$\cdots$
H	u	v		w	
$H \otimes H \otimes H$			$u\vert u\vert u$		$\cdots$

Since B has trivial higher order structure, we consider the cochain

$$\varphi = \mu\left(g_2 \otimes g - g \otimes g_2\right) + g_2\left(\mu_H \otimes \mathbf{1} - \mathbf{1} \otimes \mu_H\right),$$

which vanishes on $H^{\otimes 3}$ except $\varphi\left(u|u|u\right) = ac + ca = g\left(v\right)$; thus $\varphi = g\left(v\right)\partial_{u|u|u}$. Since $z = \mu_H\left(\mu_H \otimes \mathbf{1} - \mathbf{1} \otimes \mu_H\right) = 0$, every cocycle $b \in Hom^{-1}\left(H^{\otimes 3}, H\right)$ satisfies $\nabla b = z$. Since $v\partial_{u|u|u}$ is the only candidate, we set $b = v\partial_{u|u|u}$; then

$$\left[\tilde{g}\left(b\right) - \varphi\right] = \left[g\left(v\right)\partial_{u|u|u} - \varphi\right] = [0].$$

Choose $u = 0 \in \tilde{g}_*^{-1}\left[\tilde{g}\left(b\right) - \varphi\right]$ and define $\mu_H^3 = v\partial_{u|u|u}$; then $\mu_H^3\left(u|u|u\right) = v$. Finally, since $\varphi - g\mu_H^3 \equiv 0$; we may set $g_n = 0$ and $\mu_H^n = 0$ for all $n \geq 4$ to obtain an induced A_∞-algebra structure $\left(H, \mu_H, \mu_H^3\right)$ and a map $G = g + g_2$ of A_∞-algebras.

3. A Topological Example

Let $\mathbf{k}$ be a field. Given a space X, let $S_*\left(\Omega X; \mathbf{k}\right)$ denote the singular chains on the space of (base pointed) Moore loops on X, and choose a homology isomorphism $g : H_*\left(\Omega X; \mathbf{k}\right) \to S_*\left(\Omega X; \mathbf{k}\right)$. Since $H = H_*\left(\Omega X; \mathbf{k}\right)$ is free and $S = S_*\left(\Omega X; \mathbf{k}\right)$ is a Hopf algebra, the induced map $\tilde{g} : \mathcal{E}nd_{TH} \to U_{H,S}$ is a homology isomorphism by Proposition 1, and the Transfer Algorithm induces an A_∞-bialgebra structure on H. Let us apply this fact to a particular space X and identify a non-trivial operation $\omega_2^2 : H \otimes H \to H \otimes H$.

Given a 1-connected DGA $\left(A, d_A\right)$ over $\mathbb{Z}_2$, the bar construction of A, denoted by BA, is the cofree DGC $T^c\left(\downarrow \overline{A}\right)$ with differential d and coproduct Δ defined as follows: Let $[x_1|\cdots|x_n]$ denote the element $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n \in BA$; then

$$d\left[x_1|\cdots|x_n\right] = \sum_{i=1}^{n}\left[x_1|\cdots|dx_i|\cdots|x_n\right] + \sum_{i=1}^{n-1}\left[x_1|\cdots|x_i x_{i+1}|\cdots|x_n\right];$$

$$\Delta\left[x_1|\cdots|x_n\right] = [\ ] \otimes \left[x_1|\cdots|x_n\right] + \left[x_1|\cdots|x_n\right] \otimes [\ ] + \sum_{i=1}^{n-1}\left[x_1|\cdots|x_i\right] \otimes \left[x_i|\cdots|x_n\right].$$

Consider the space $Y = \left(S^2 \times S^3\right) \vee \Sigma\mathbb{C}P^2$, multiplicative generators $\bar{a}_i \in H^i\left(S^i; \mathbb{Z}_2\right)$, $\bar{b} \in H^3\left(\Sigma\mathbb{C}P^2; \mathbb{Z}_2\right)$ and $Sq^2\bar{b} \in H^5\left(\Sigma\mathbb{C}P^2; \mathbb{Z}_2\right)$, a map $f : Y \to K(\mathbb{Z}_2, 5)$ such that $f^*(\iota_5) = \bar{a}_2\bar{a}_3 + Sq^2\bar{b}$, and the pullback $p : X \to Y$ of the following path fibration:

$$\begin{array}{ccccc}
K\left(\mathbb{Z}_2, 4\right) & \longrightarrow & X & \longrightarrow & \mathcal{L}K\left(\mathbb{Z}_2, 5\right) \\
 & & p\downarrow & & \downarrow \\
 & & Y & \xrightarrow{f} & K\left(\mathbb{Z}_2, 5\right). \\
 & & \bar{a}_2\bar{a}_3 + Sq^2\bar{b} & \xleftarrow[f^*]{} & \iota_5
\end{array}$$

Let $a_i = p^*(\bar{a}_i)$ and $b = p^*(\bar{b})$; then $A = H^*(X; \mathbb{Z}_2) = \left\{1, a_2, a_3, b, a_2 a_3 = Sq^2 b, \ldots\right\}$.

Form the bar construction BA; since $H = H^*\left(BA\right) \approx H^*\left(\Omega X; \mathbb{Z}_2\right)$ as coalgebras, $\left(BA, d, \Delta\right)$ is a DG coalgebra model for cochains on ΩX. In [1], H.-J. Baues identified a compatible multiplication $\mu : BA \otimes BA \to BA$ and a DG Hopf algebra model $\left(BA, d, \Delta, \mu\right)$ in the following way: The twisting in X induces Steenrod's

$\smile_1 \colon A \otimes A \to A$, which acts non-trivially via $b \smile_1 b = a_2 a_3$ and the induced map $\phi : BA \otimes BA \to A$ acts non-trivially via

$$\phi\left([x] \otimes [\ ]\right) = \phi\left([\ ] \otimes [x]\right) = x \text{ and } \phi\left([b] \otimes [b]\right) = b \smile_1 b = a_2 a_3.$$

(cf. [2], [4]). Consider the tensor product of coalgebras $BA \otimes BA$ with coproduct $\psi = \sigma_{2,2}\left(\Delta \otimes \Delta\right)$ and define

$$\mu := \sum\nolimits_{k \geq 0} \underbrace{\left(\downarrow \phi \otimes \cdots \otimes \downarrow \phi\right)}_{k+1 \text{ factors}} \bar{\psi}^{(k)},$$

where $\bar{\psi}^{(0)} = \mathbf{1}$, $\bar{\psi}^{(k)} = \left(\bar{\psi} \otimes \mathbf{1}^{\otimes k-1}\right) \cdots \left(\bar{\psi} \otimes \mathbf{1}\right) \bar{\psi}$ for $k > 0$, and $\bar{\psi}$ is the reduced coproduct. Then for example, $\mu\left([b] \otimes [b]\right) = [a_2 a_3]$.

Let μ_H be the multiplication on H induced by μ and consider the classes $\alpha_i = \operatorname{cls}[a_i], \beta = \operatorname{cls}[b] \in H$. Choose a cocycle-selecting map $g : H \to BA$ such that $g\left(\operatorname{cls}[x_1|\cdots|x_n]\right) = [x_1|\cdots|x_n]$. Then $\mu\left([b] \otimes [b]\right) = [a_2 a_3] = d\,[a_2|a_3]$ implies $\mu_H\left(\beta \otimes \beta\right) = 0$ and $\left(g\mu_H + \mu\left(g \otimes g\right)\right)\left(\beta \otimes \beta\right) = [a_2 a_3]$. Nevertheless, by the Transfer Theorem, there is a cochain homotopy $g_2^1 : H \otimes H \to BA$ satisfying the relation $\nabla g_2^1 = g\mu_H + \mu\left(g \otimes g\right)$ such that $g_2^1\left(\beta \otimes \beta\right) = [a_i|a_{5-i}]$ for some $i \in \{2, 3\}$; and in particular, we may choose

$$g_2^1\left(\beta \otimes \beta\right) = [a_2|a_3]$$

since either choice gives rise to isomorphic structures. Let Δ_H be the coproduct induced by Δ; then $\{\Delta g + \left(g \otimes g\right)\Delta_H\}\left(\beta\right) = 0$ since β is primitive. By the Transfer Theorem, there is a cochain homotopy $g_1^2 : H \to BA \otimes BA$ satisfying the relation $\nabla g_1^2 = \Delta g + \left(g \otimes g\right)\Delta_H$ such that $\nabla g_1^2\left(\beta\right) = 0$. Thus $g_1^2\left(\beta\right) = \lambda \otimes [a_2] + [a_2] \otimes \rho$ for some $\lambda, \rho \in \mathbb{Z}_2$; and in particular, we may choose

$$g_1^2\left(\beta\right) = 0.$$

Again by the Transfer Theorem, there is a cochain homotopy $g_2^2 : H \otimes H \to BA \otimes BA$ satisfying the following relation on $JJ_{2,2}$:

$$\begin{aligned}\nabla g_2^2 &= \left(\mu \otimes \mu\right)\sigma_{2,2}\left(\Delta g \otimes g_1^2 + g_1^2 \otimes \left(g \otimes g\right)\Delta_H\right) \\ &\quad + \left(\mu\left(g \otimes g\right) \otimes g_2^1 + g_2^1 \otimes g\mu_H\right)\sigma_{2,2}\left(\Delta_H \otimes \Delta_H\right) \\ &\quad + \omega_{BA}^{2,2}\left(g \otimes g\right) + \left(g \otimes g\right)\omega_H^{2,2} + \Delta g_2^1 + g_1^2 \mu_H.\end{aligned} \tag{3.1}$$

The component $\omega_{BA}^{2,2}\left(g \otimes g\right)$ vanishes since BA has trivial higher order structure; the non-triviality of $\left(g \otimes g\right)\omega_H^{2,2}$ is to be determined.

Let us evaluate relation (3.1) at $\beta \otimes \beta$. First, $g_1^2 \mu_H\left(\beta \otimes \beta\right) = 0$ by the observation above, and $\left(\mu \otimes \mu\right)\sigma_{2,2}\left(\Delta g \otimes g_1^2 + g_1^2 \otimes \left(g \otimes g\right)\Delta_H\right)\left(\beta \otimes \beta\right) = 0$ by our choice of g_1^2. Second, $\left(\mu\left(g \otimes g\right) \otimes g_2^1 + g_2^1 \otimes g\mu_H\right)\sigma_{2,2}\left(\Delta_H \otimes \Delta_H\right)\left(\beta \otimes \beta\right) = [\ ] \otimes g_2^1\left(\beta \otimes \beta\right) + g_2^1\left(\beta \otimes \beta\right) \otimes [\ ] = \left(\Delta + \overline{\Delta}\right) g_2^1\left(\beta \otimes \beta\right)$. Thus relation (3.1) reduces to $\nabla g_2^2(\beta \otimes \beta) = \left(g \otimes g\right)\omega_H^{2,2}\left(\beta \otimes \beta\right) + \overline{\Delta} g_2^1\left(\beta \otimes \beta\right) = \left(g \otimes g\right)\omega_H^{2,2}\left(\beta \otimes \beta\right) + [a_2] \otimes [a_3]$, and we conclude that

$$\omega_H^{2,2}\left(\beta \otimes \beta\right) = \alpha_2 \otimes \alpha_3.$$

Thus the topology of the total space X in the fibration $p : X \to \left(S^2 \times S^3\right) \vee \Sigma\mathbb{C}P^2$ above induces a nontrivial 2-in/2-out operation $\omega_H^{2,2}$ on $H = H^*\left(\Omega X; \mathbb{Z}_2\right)$. A variation of this example, with a nontrivial topologically induced 2-in/n-out operation on loop cohomology for each $n \geq 2$, appears in [7].

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